Construction of Balanced Bipartite Block Designs with Unequal Block Sizes

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ABSTRACT

In this paper some methods of construction of balanced bipartite block (BBPB) designs are introduced for comparison of two disjoint sets of treatments (test treatments and control treatments). The methods are based on incidence matrices of balanced incomplete block (BIB) designs. The derived results are given with examples.

1. Introduction

Many researchers have contributed in the construction of incomplete block designs for comparison of test-control treatments by understanding their significance. Bechhofer and Tamhane (1981) introduced balanced treatment incomplete block (BTIB) designs. Kageyama and Sinha (1988) (see also Sinha and Kageyama (1990)) defined balanced bipartite block (BBPB) designs as an extension of these BTIB designs and gave some systematic methods of construction of BIB designs. Majumdar (1986) obtained sufficient conditions for a block design to be A-optimal for comparison of a set of test treatments to a set of control treatments. Angelis and Moyssiadis (1991) (see also Angelis, Moyssiadis and Kageyama (1993)) gave balanced treatment incomplete block designs with unequal block sizes (BTIUB) for comparison of a set of test treatments to a single control treatment with unequal blocks. The more results have been studied by several researchers (see e.g. Jacroux (1992, 2000, 2002), Parsad and Gupta (1994), Parsad, Gupta and Singh (1996), Gupta and Parsad (2001)) for the comparison of test treatments and control treatments.

Here the problem of deriving an incomplete block design is considered for comparing \(v_1\) test treatments with \(v_2 (\geq 2)\) control treatments. The \(v_1\) test treatments are denoted by \(1, 2, \ldots, v_1\) and \(v_2\) control treatments are denoted by \(v_1 + 1, \ldots, v_1 + v_2 (= v)\). Consider an incomplete block design \(D (v = v_1 + v_2), b = \sum_{i=1}^{p} b_i, r' = (r_1, r_2, \ldots, r_p), k = (k_1, k_2, \ldots, k_p)\) whose incidence matrix \(N = (n_{ij})\) is of order \(v (= v_1 + v_2) \times d\). Then a connected block design is called a balanced bipartite block design with unequal block sizes (BBPBUB) of Jaggi, Parsad and Gupta (1999), if the information matrix for treatment effects i.e. the \(I\) matrix of the design is given by

\[
I = \begin{pmatrix}
(a_1 + s_1)I_{v_2} - s_1I_{v_2} & -s_0J_{v_2}v_3 \\
-s_0J_{v_2}v_3 & (a_2 + s_2)I_{v_3} - s_2I_{v_3}
\end{pmatrix}
\]

where \(s_1, s_0, \) and \(s_2\) are off-diagonal elements and \(a_1\) and \(a_2\) are some scalar constants such that \(a_1 = (v_1 - 1)s_1 - v_2s_0 = 0\) and \(a_2 = (v_2 - 1)s_2 - v_1s_0 = 0\). For a binary design, the constants \(a_1\) and \(a_2\) represent the functions of replications of the two sets of treatments and block sizes in the design.

In following section, we obtain some methods of construction of BIB designs for comparison of a set of test treatments to a set of control treatments by using BIB designs. The definition of BIB design is given in Dey (2010).

In what follows, we denote by \(\otimes\) the kronecker product of matrices, \(O_{x \times y}\) the null matrix of order \(x \times y\), \(I_x\) the identity matrix of order \(x, J_{x \times y}\) the matrix of ones of order \(x \times y\), \(1_x\) the \(1 \times x\) row vector of ones \(1_x \otimes N\) the \(x\) replications of \(N\) and by \(p_1, p_2, p_3, p_4, p_5\) the positive integers.

2. Methods of construction of BIB Designs

Using the incidence matrices of BIB designs, etc. we describe below some methods of construction of BIB designs.
Let $N_i (L = 1,2,3,4,5)$ be the $v_i \times b_i$ incidence matrix of a BIB design with parameters $v_i, b_i, r_i, k_i, \lambda_i$ such that $v_2 = v_4, v_3 = v_5$ and $v_1 = v_2 + v_3$. Now we form the matrix $N$ as

$$N = \begin{bmatrix}
1_p' \otimes N_1 & 1_p' \otimes N_2 & J_{v_2 \times p_3 b_3} & 1_p' \otimes N_4 & O_{v_2 \times p_2 b_5} & I_{v_2} & O_{v_2 \times v_3} \\
1_p' \otimes N_3 & 1_p' \otimes N_5 & O_{v_3 \times p_4 b_4} & 1_p' \otimes N_5 & O_{v_3 \times v_5} & I_{v_3}
\end{bmatrix} \quad (2.1)
$$

**Theorem 1**: Block design with the incidence matrix $N$ of the form (2.1) is the BBPB design $D$ with unequal block sizes with parameters $v_i' = v_2, v_3' = v_3, b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + p_5 b_5 + v_4 + v_5, r = \{(p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4 + 1) 1_p, (p_1 r_1 + p_2 r_2 + p_3 r_3 + p_5 r_5 + 1) 1_p\} = \{k_1 1_p b_1, (k_2 + v_3) 1_p b_2, (k_3 + v_5) 1_p b_5, k_4 1_p b_4, k_5 1_p b_5\}.

**Proof**: For the block design with incidence matrix $N$ given in (2.1) we have $C$ matrix as in (1.1).

$$C = \begin{bmatrix}
(a_1 + s_1) I_{v_2} - s_1 I_{v_2} & -s_0 J_{v_2 \times v_3} \\
-s_0 J_{v_3 \times v_2} & (a_2 + s_2) I_{v_3} - s_2 I_{v_2}
\end{bmatrix}
$$

The diagonal and off-diagonal elements of $C (= c_{ij})$ matrix are respectively given by

$$c_{ij} = a_1 = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_4 r_4 (k_4 - 1)}{k_4}
$$

$$c_{ij} = a_2 = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_5 r_5 (k_5 - 1)}{k_5}
$$

and

$$c_{ij} = s_1 = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 \lambda_2}{(k_2 + v_3)} + \frac{p_3 \lambda_3}{(k_3 + v_2)} + \frac{p_4 \lambda_4}{k_4}
$$

$$c_{ij} = s_0 = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 r_2}{(k_2 + v_3)} + \frac{p_5 r_5}{(k_5 + v_3)}
$$

$$c_{ij} = s_2 = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 b_2}{(k_2 + v_3)} + \frac{p_3 \lambda_3}{(k_3 + v_2)} + \frac{p_5 \lambda_5}{k_5}
$$

Then by Jaggi, Parsad and Gupta (1999),

$$\frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_4 r_4 (k_4 - 1)}{k_4}
$$

$$- (v_2 - 1) \left\{ \frac{p_1 \lambda_1}{k_1} + \frac{p_2 \lambda_2}{(k_2 + v_3)} + \frac{p_3 \lambda_3}{(k_3 + v_2)} + \frac{p_4 \lambda_4}{k_4} \right\} - v_3 \left\{ \frac{p_1 \lambda_1}{k_1} + \frac{p_2 r_2}{(k_2 + v_3)} + \frac{p_5 r_5}{(k_5 + v_3)} \right\}
$$

$$= 0$$

and

$$\frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 b_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_5 r_5 (k_5 - 1)}{k_5}
$$

$$- (v_3 - 1) \left\{ \frac{p_1 \lambda_1}{k_1} + \frac{p_2 b_2}{(k_2 + v_3)} + \frac{p_3 \lambda_3}{(k_3 + v_2)} + \frac{p_5 \lambda_5}{k_5} \right\} - v_2 \left\{ \frac{p_1 \lambda_1}{k_1} + \frac{p_2 b_2}{(k_2 + v_3)} + \frac{p_5 \lambda_5}{(k_5 + v_3)} \right\}
$$

$$= 0.$$

Hence the proof.

**Example 2.1**: Consider five BIB designs with parameters $(9,12,4,3,1), (5,10,6,3,3), (4,6,3,2,1), (5,10,4,2,1)$ and $(4,4,3,3,2)$ respectively. Then taking $p_1 = p_2 = p_3 = p_4 = 1$ and $p_5 = 2$, the design $D$ with incidence matrix $N$ as in (2.1) is a non-proper
non-equipartite BBPB design with parameters \(v_1^* = 5, v_2^* = 4, b = 55, r' = \{211_4, 241_4\}, k' = \{31_{12}, 71_{10}, 71_{10}, 21_{10}, 31_6, 11_5, 11_5\}\).

**Corollary 2.1:** In Theorem 2.1, if we remove last \(v_2\) and \(v_3\) blocks, then we get a BBPB design \(D\) with unequal block sizes with parameters \(v_1' = v_2 = v_3, b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + p_5 b_5, r' = \{(p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4 + p_5 r_5) 1_{p_1 b_1}, (k_2 + v_3) 1_{p_2 b_3}, (k_3 + v_2) 1_{p_3 b_5}, k_4 1_{p_4 b_4}, k_5 1_{p_5 b_5}\}\).

**Example 2.2:** In Example 2.1, if we remove last \(v_2\) and \(v_3\) blocks, then we get a non-proper non-equipartite BBPB design \(D\) with \(p_1 = p_2 = p_3 = p_4 = 1\) and \(p_5 = 2\). The parameters of the design are \(v_1^* = 5, v_2^* = 4, b = 46, r' = \{201_4, 231_4\}, k' = \{31_{12}, 71_{10}, 71_{10}, 21_{10}, 31_6\}\).

**Corollary 2.2:** In Theorem 2.1, if we remove last \(p_1 b_4, p_2 b_5, v_2\) and \(v_3\) blocks, then again we get a BBPB design \(D\) with unequal block sizes with parameters \(v_1' = v_2, v_2' = v_3, b = p_1 b_1 + p_2 b_2 + p_3 b_3, r' = \{(p_1 r_1 + p_2 r_2 + p_3 b_3) 1_{v_2}, (p_1 r_1 + p_2 b_2 + p_3 r_3) 1_{v_3}\}\).

**Example 2.3:** Consider three BIB designs with parameters \((9,12,4,3,1)\), \((5,10,4,2,1)\) and \((4,6,3,2,1)\) respectively. Then using Corollary 2.2 and taking \(p_1 = 1, p_2 = 3\) and \(p_3 = 2\), we get a non-proper non-equipartite BBPB design \(D\) with parameters \(v_1^* = 5, v_2^* = 4, b = 50, r' = \{241_5, 401_4\}, k' = \{31_{12}, 61_{20}, 81_{18}\}\).

**Remark 2.1:** In Corollary 2.2 if \(p_2 = k_2 = k_3 = k\) and \(p_3 = k_4\), then we get a BBPB design \(D\) with unequal block sizes with parameters \(v_1' = v_2, v_2' = v_3, b = p_1 b_1 + kb_2 + k b_3, r' = \{(p_1 r_1 + k r_2 + k b_3) 1_{v_2}, (p_1 r_1 + k b_2 + k r_3) 1_{v_3}\}\).

**Example 2.4:** Consider three BIB designs with parameters \((9,12,4,3,1)\), \((5,10,4,2,1)\) and \((4,6,3,2,1)\) respectively. Then taking \(p_1 = 1, p_2 = 2\) and \(p_3 = 3\), we get a non-proper non-equipartite BBPB design \(D\) with parameters \(v_1^* = 5, v_2^* = 4, b = 50, r' = \{301_5, 331_4\}, k' = \{31_{12}, 61_{20}, 71_{16}\}\).

**Remark 2.2:** A special case of Corollary 2.2 arise when \(p_1 = p_2 = p_3 = 1\). Then the resulting design is again a BBPB design \(D\) with unequal block sizes with parameters \(v_1' = v_2, v_2' = v_3, b = b_1 + b_2 + b_3, r' = \{(r_1 + r_2 + b_3) 1_{v_2}, (r_1 + b_2 + r_3) 1_{v_3}\}\).

**Example 2.5:** Consider three BIB designs with parameters \((8,14,7,4,3)\), \((5,10,4,2,1)\) and \((3,3,2,2,1)\) respectively. Then we get a non-proper non-equipartite BBPB design \(D\) with parameters \(v_1^* = 5, v_2^* = 3, b = 27, r' = \{141_5, 191_3\}, k' = \{411_4, 51_{10}, 71_3\}\).

**Remark 2.3:** Following theorems can be proved on the similar lines of Theorem 2.1. So we avoided proofs of the Theorems.

Let \(N_L = (L = 1,2,3,4,5)\) be the \(v_1 \times b_1\) incidence matrix of a BIB design with parameters \(b_L, l_L, k_L, l_L, k_L, l_L\) such that \(v_2 = v_4, v_3 = v_5\) and \(v_1 = v_2 + v_3\). Now we form the matrix \(N\) as

\[
N = \begin{bmatrix}
1_{b_1} \otimes N_1 & 1_{b_2} \otimes N_2 & I_{b_2 \times p_2 b_2} & 1_{b_3} \otimes N_4 & I_{b_3 \times p_3 b_3} & I_{b_4 \times p_4 b_4} & I_{b_5 \otimes N_5} & I_{b_5} & O_{b_5 \times v_3} & O_{b_5 \times v_3} & I_{v_2} & O_{b_5 \times v_3} & I_{v_3} \end{bmatrix}
\]  

(2.2)

**Theorem 2.2:** Block design with the incidence matrix \(N\) of the form (2.2) is the BBPB design \(D\) with unequal block sizes with parameters \(v_1' = v_2, v_2' = v_3, b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + p_5 b_5 + v_2 + v_3, r' = \{(p_1 r_1 + p_2 r_2 + p_3 b_3 + p_4 b_4 + p_5 r_3 + 1) 1_{v_2}, (p_1 r_2 + p_2 r_2 + p_3 r_3 + p_4 r_4 + p_5 r_3 + 1) 1_{v_3}\}\).

\[
k' = (k_1 1_{p_1 b_1}, (k_2 + v_2) 1_{p_2 b_2}, (k_3 + v_3) 1_{p_3 b_3}, (k_4 + v_2) 1_{p_4 b_4}, (k_5 + v_3) 1_{p_5 b_5}, 11_{v_2}, 11_{v_3})
\]

The diagonal and off-diagonal elements of its \(C(= \text{matrix})\) are respectively given by

\[
c_{ij} = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 b_1 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_4 r_4 (k_4 + v_3 - 1)}{(k_4 + v_3)} + \frac{p_5 b_5 (k_5 + v_2 - 1)}{(k_5 + v_2)}
\]

\[
c_{ij} = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 b_2 (k_2 + v_3 - 1)}{(k_2 + v_3)} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{p_4 b_4 (k_4 + v_3 - 1)}{(k_4 + v_3)} + \frac{p_5 r_5 (k_5 + v_2 - 1)}{(k_5 + v_2)}
\]

and
Corollary 2.3: In Theorem 2.2, if we remove last $v_2$ and $v_3$ blocks, then we get a BBPB design $D$ with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + p_5 b_5$, \( r' = \{(p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4 + p_5 r_5)1_{v_1} (p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4 + p_5 r_5)1_{v_3}\}$, \( k = \{k_1 1_{p_1 b_1} (k_2 + v_2) 1_{p_2 b_2} (k_3 + v_3) 1_{p_3 b_3} (k_4 + v_4) 1_{p_4 b_4} (k_5 + v_5) 1_{p_5 b_5}\} \).

Example 2.7: In Example 2.6, if we remove last $v_2$ and $v_3$ blocks, then we get a non-proper non-equireplicate BBPB design $D$ with $p_1 = p_2 = p_3 = p_4 = p_5 = 1$. The parameters of the design are $v_1^* = 7$, $v_2^* = 4$, $b = 35$, \( r' = \{(221_1'_1, 254_1\} \), \( k' = \{51_1'_1, 81_1_5, 91_1_6, 71_1_7, 101_4, 117_1_1\} \).

Let $N_i (L = 1, 2, 3, 4, 5)$ be the $v_i \times b_i$ incidence matrix of a BIB design with parameters $v_i, b_i, r_i, k_i, \lambda_i$ such that $v_2 = v_4$, $v_3 = v_5$ and $v_1 = v_2 + v_3$. Now we form the matrix $N$ as

\[
N = \begin{bmatrix}
1_{p_1} \otimes N_1 & 1_{p_2} \otimes N_2 & I_{v_2} \otimes N_3 & I_{v_3} \otimes N_2 & I_{v_3} \otimes N_3 & O_{v_2 \times v_3} & I_{v_3} \otimes N_3
\end{bmatrix} \tag{2.3}
\]

Theorem 2.3: Block design with the incidence matrix $N$ of the form (2.3) is the BBPB design $D$ with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + p_5 b_5 + v_2 + v_3$, \( r' = \{(p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4 + p_5 r_5 + 1)\} \), \( k = \{k_1 1_{p_1 b_1} (k_2 + v_2) 1_{p_2 b_2} (k_3 + v_3) 1_{p_3 b_3} (k_4 + v_4) 1_{p_4 b_4} (k_5 + v_5) 1_{p_5 b_5}\} \). The diagonal and off-diagonal elements of its $C(=c_{ij})$ matrix are respectively given by

\[
c_{ij} = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{k_2 + v_3} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{k_3 + v_2} + \frac{p_4 r_4 (k_4 + v_3 - 1)}{k_4 + v_3} + \frac{b k_5}{(k_5 + 1)}.
\]

\[
c_{ij} = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{k_2 + v_3} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{k_3 + v_2} + \frac{p_4 r_4 (k_4 + v_3 - 1)}{k_4 + v_3} + \frac{v_5 k_3}{(k_5 + 1)}
\]

and

\[
c_{ij} = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 r_2 (k_2 + v_3 - 1)}{k_2 + v_3} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{k_3 + v_2} + \frac{p_4 r_4 (k_4 + v_3 - 1)}{k_4 + v_3} + \frac{b k_5}{(k_5 + 1)}.
\]

Example 2.8: Consider five BIB designs with parameters $(11, 11, 5, 5, 2), (6, 15, 5, 2, 1), (5, 10, 6, 3, 3), (6, 6, 5, 5, 4)$ and $(5, 5, 4, 4, 3)$ respectively. Then taking $p_1 = p_2 = p_3 = p_4 = 1$, the design $D$ with incidence matrix $N$ as in (2.3) is a non-proper non-equireplicate BBPB design with parameters $v_1^* = 6$, $v_2^* = 5$, $b = 83$, \( r' = \{311_6, 571_5\} \), \( k' = \{51_1'_1, 71_1_5, 91_1_6, 101_5, 51_3_3, 117_1_1\} \).

Corollary 2.4: In Theorem 2.4, if we remove last $v_2$ and $v_3$ blocks, then we get a BBPB design $D$ with unequal block sizes with parameters $v_1^* = v_2$, $v_2^* = v_3$, $b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + v_2 + v_3$, \( r' = \{(p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4 + b_5)\} \), \( k = \{k_1 1_{p_1 b_1} (k_2 + v_2) 1_{p_2 b_2} (k_3 + v_3) 1_{p_3 b_3} (k_4 + v_4) 1_{p_4 b_4} (k_5 + v_5) 1_{v_2 b_5}\} \).
Example 2.9: In Example 2.8, if we remove last \(v_2 \) and \(v_3 \) blocks, then we get a non-proper non-equireplicate BBPB design \(D \) with \( p_1 = p_2 = p_3 = p_4 = 1 \). The parameters of the design are \( \nu_1^* = 6 \), \( \nu_2^* = 5 \), \( b = 72 \), \( r^* = \{301, 561, 5 \} \), \( k^* = \{51, 17, 9, 101, 0, 5130\} \).

Let \( N_i (L = 1, 2, 3, 4, 5) \) be the \( (\nu_i \times b_i) \) incidence matrix of a BIB design with parameters \( \nu_i, b_i, r_i, k_i, \lambda_i \) such that \( v_2 = v_4, v_3 = v_5 \) and \( v_1 = v_2 + v_3 \). Now we form the matrix \( N \) as

\[
N = \begin{bmatrix}
N_1 & \cdots & N_5 \\
N_2 & \cdots & N_5 \\
N_3 & \cdots & N_5 \\
N_4 & \cdots & N_5 \\
N_5 & \cdots & N_5
\end{bmatrix}
\]

(2.4)

Theorem 2.4: Block design with the incidence matrix \( N \) of the form (2.4) is the BBPB design \( D \) with unequal block sizes with parameters \( \nu_1 = v_2, \nu_2 = v_3, b = p_1b_1 + p_2b_2 + p_3b_3 + p_4b_4 + p_5b_5 + v_2 + v_3, r^* = \{(p_1r_1 + p_2r_2 + p_3r_3 + p_4r_4 + p_5r_5 + 1)1_{v_2}, (p_1r_1 + p_2r_2 + p_3r_3 + p_5r_5 + 1)1_{v_5}\} \), \( k^* = \{s, (\nu_1 + v_2 + v_3)1_{p_1b_1}, (s + v_2)1_{p_2b_2}, (s + v_3)1_{p_3b_3}, (s + v_5)1_{p_5b_5}\} \). The diagonal and off-diagonal elements of its \((c_{ij})\) matrix are respectively given by

\[
c_{ij} = \begin{cases}
p_1r_1(k_1 - 1) & \text{if } i = j, i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2, 3, 4, 5, \\
p_2r_2(k_2 + v_2 - 1) & \text{if } i = j, i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2, 3, 4, 5, \\
p_3r_3(k_3 + v_3 - 1) & \text{if } i = j, i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2, 3, 4, 5, \\
p_4r_4(k_4 - 1) & \text{if } i = j, i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2, 3, 4, 5, \\
p_5r_5(k_5 + v_5 - 1) & \text{if } i = j, i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2, 3, 4, 5.
\end{cases}
\]

Example 2.10: Consider five BIB designs with parameters \((9,12,4,3,1), (5,5,4,4,4), (4,4,3,3,2), (5,10,6,3,3)\) and \((4,6,3,2,1)\) respectively. Then taking \( p_1 = p_2 = p_3 = p_4 = 2 \), the design \( D \) with incidence matrix \( N \) as in (2.4) is a non-proper non-equireplicate BBPB design with parameters \( \nu_i^* = 5, \nu_j^* = 4, b = 61, r^* = \{391, 271\} \), \( k^* = \{31, 12, 38, 31, 0, 310, 71, 11, 111\} \).

Corollary 2.5: In Theorem 2.4, if we remove last \( v_2 \) and \( v_3 \) blocks, then we get a BBPB design \( D \) with unequal block sizes with parameters \( \nu_1 = v_2, \nu_2 = v_3, b = p_1b_1 + p_2b_2 + p_3b_3 + p_4b_4 + p_5b_5 + v_2 + v_3, r^* = \{(p_1r_1 + p_2r_2 + p_3r_3 + p_4r_4 + p_5r_5 + 1)1_{v_2}, (p_1r_1 + p_2r_2 + p_3r_3 + p_5r_5 + 1)1_{v_5}\} \), \( k^* = \{s, (s + v_2)1_{p_1b_1}, (s + v_3)1_{p_2b_2}, (s + v_5)1_{p_3b_3}, (s + v_5)1_{p_5b_5}\} \).

Example 2.11: In Example 2.10, if we remove last \( v_2 \) and \( v_3 \) blocks, then we get a non-proper non-equireplicate BBPB design \( D \) with \( p_1 = p_2 = p_3 = 1 \) and \( p_4 = p_5 = 2 \). The parameters of the design are \( \nu_i^* = 5, \nu_j^* = 4, b = 61, r^* = \{381, 261\} \), \( k^* = \{31, 12, 38, 31, 0, 310, 71, 11, 111\} \).

Let \( N_i (L = 1, 2, 3, 4, 5) \) be the \( (\nu_i \times b_i) \) incidence matrix of a BIB design with parameters \( \nu_i, b_i, r_i, k_i, \lambda_i \) such that \( v_2 = v_4, v_3 = v_5 \) and \( v_1 = v_2 + v_3 \). Now we form the matrix \( N \) as

\[
N = \begin{bmatrix}
N_1 & \cdots & N_5 \\
N_2 & \cdots & N_5 \\
N_3 & \cdots & N_5 \\
N_4 & \cdots & N_5 \\
N_5 & \cdots & N_5
\end{bmatrix}
\]

(2.5)

Theorem 2.5: Block design with the incidence matrix \( N \) of the form (2.5) is the BBPB design \( D \) with unequal block sizes with parameters \( \nu_1 = v_2, \nu_2 = v_3, b = p_1b_1 + p_2b_2 + p_3b_3 + p_4b_4 + v_2 + v_3, r^* = \{(p_1r_1 + p_2r_2 + p_3r_3 + p_4r_4 + b_3 + 1)1_{v_2}, (p_1r_1 + p_2r_2 + p_3r_3 + p_5r_5 + 1)1_{v_5}\} \), \( k^* = \{s, (s + v_2)1_{p_1b_1}, (s + v_3)1_{p_2b_2}, (s + v_5)1_{p_3b_3}, (s + v_5)1_{p_5b_5}\} \). The diagonal and off-diagonal elements of its \((c_{ij})\) matrix are respectively given by

\[
c_{ij} = \begin{cases}
p_1r_1(k_1 - 1) & \text{if } i = j, i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2, 3, 4, 5, \\
p_2r_2(k_2 + v_2 - 1) & \text{if } i = j, i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2, 3, 4, 5, \\
p_3r_3(k_3 + v_3 - 1) & \text{if } i = j, i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2, 3, 4, 5, \\
p_4r_4(k_4 - 1) & \text{if } i = j, i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2, 3, 4, 5, \\
\end{cases}
\]
\[ c_{ij} = \frac{p_1 r_1 (k_1 - 1)}{k_1} + \frac{p_2 b_2 (k_2 - v_2 - 1)}{(k_2 + v_3)} + \frac{p_3 r_3 (k_3 + v_2 - 1)}{(k_3 + v_2)} + \frac{v_2 r_5 k_5}{(k_5 + 1)} \]

and

\[ c_{ij} = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 \lambda_2}{(k_2 + v_3)} + \frac{p_3 b_3}{(k_3 + v_2)} + \frac{p_4 \lambda_4}{k_4}, \]

\[ c_{ij} = \frac{p_1 r_1}{k_1} + \frac{p_2 r_2}{(k_2 + v_3)} + \frac{p_3 r_3}{(k_3 + v_2)} + \frac{r_5}{(k_5 + 1)}, \]

\[ c_{ij} = \frac{p_1 \lambda_1}{k_1} + \frac{p_2 b_2}{(k_2 + v_3)} + \frac{p_3 \lambda_3}{(k_3 + v_2)} + \frac{v_2 r_5}{(k_5 + 1)}. \]

**Example 2.12:** Consider five BIB designs with parameters (11,11,5,5,2), (7,7,4,4,2), (4,4,3,3,2), (7,7,3,3,1) and (4,6,3,2,1) respectively. Then taking \( p_1 = p_2 = p_4 = 1 \) and \( p_3 = 4 \), the design \( D \) with incidence matrix \( N \) as in (2.5) is a non-equireplicate BBPB design with parameters \( v_1^* = 7, v_2^* = 4, b = 94, r^* = (35, 45, 4), k^* = [51, 81, 101, 31, 31], 11, 11]. \)

**Corollary 2.6:** In Theorem 2.5, if we remove last \( v_2 \) and \( v_3 \) blocks, then we get a BBPB design \( D \) with unequal block sizes with parameters \( v_1^* = v_2, v_2^* = v_3, b = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 + v_2 b_5, r^* = (p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4 + b_5) \).

**Example 2.13:** In Example 2.12, if we remove last \( v_2 \) and \( v_3 \) blocks, then we get a non-equireplicate BBPB design \( D \) with \( p_1 = p_2 = p_4 = 1 \) and \( p_3 = 4 \). The parameters of the design are \( v_1^* = 7, v_2^* = 4, b = 83, r^* = \{341, 441\}, k^* = [51, 81, 101, 31, 31, 31, 31]. \)

3. Conclusion

Here for comparison of a set of test treatments to a set of control treatments balanced bipartite block designs with unequal block sizes are derived by the new methods of construction. Such designs can be applied in pharmaceutical, industrial and agricultural experiments. The methods are flexible enough to incorporate number of incidence matrices of BIB designs.

References