Status Sum Adjacency Energy of Graphs

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ABSTRACT

The status \( \sigma(u) \) of a point \( u \) in a connected graph \( G \) is the sum of the distances between \( u \) and all other vertices in \( G \). The status sum adjacency matrix of a graph \( G \) is defined as \( S_a(G) = [s_{ij}] \), where \( s_{ij} = \sigma(v_i) + \sigma(v_j) \), if \( v_i \) is adjacent to \( v_j \) and \( s_{ij} = 0 \), otherwise. The status sum adjacency energy is defined as the sum of the absolute values of the eigenvalues of \( S_a(G) \). In this paper we obtain bounds for the status sum adjacency energy of a graph.

1. Introduction

Let \( G \) be a simple, connected graph with \( n \) vertices and \( m \) edges. Let \( V(G) \) be the vertex set of \( G \) and \( E(G) \) be an edge set of \( G \). The edge joining the vertices \( u \) and \( v \) is denoted by \( uv \). The degree of a vertex \( u \) in \( G \) is the number of edges adjacent to it and is denoted by \( d(u) \). The distance between two vertices \( u \) and \( v \), denoted by \( d(u,v) \) is the length of shortest path joining them. The diameter of a graph \( G \), denoted by \( \text{diam}(G) \), is the maximum distance between any pair of vertices of \( G \). For graph theoretic terminology we refer the book [1].

The adjacency matrix of a graph \( G \) is an \( n \times n \) matrix \( A(G) = [a_{ij}] \), in which \( a_{ij} = 1 \) if the vertices \( v_i \) and \( v_j \) are adjacent and \( a_{ij} = 0 \), otherwise.

The status of a vertex \( u \) in a connected graph \( G \) is defined as [2]

\[
\sigma(u) = \sum_{v \in V(G)} d(u,v),
\]

where \( d(u,v) \) is the distance between the vertices \( u \) and \( v \) in \( G \) and \( V(G) \) is the vertex set of \( G \).

The status sum adjacency matrix of a connected graph \( G \) is defined as [3] \( S_a(G) = [s_{ij}] \), where \( s_{ij} = \sigma(v_i) + \sigma(v_j) \) if \( v_i \) and \( v_j \) are adjacent and \( s_{ij} = 0 \), otherwise. Let the eigenvalues of \( S_a(G) \) be denoted by \( x_1, x_2, \ldots, x_n \). As \( S_a(G) \) is a real symmetric matrix, its eigenvalues are real.

The first status connectivity index \( S_1(G) \) and second status connectivity index \( S_2(G) \) of a connected graph \( G \) are defined as [2]

\[
S_1(G) = \sum_{uv \in E(G)} [\sigma(u) + \sigma(v)]
\]

and

\[
S_2(G) = \sum_{uv \in E(G)} \sigma(u)\sigma(v)
\]

where \( E(G) \) is an edge set of \( G \). The first and second Zagreb indices of \( G \) are defined as [4]

\[
Z_1(G) = \sum_{uv \in E(G)} [d(u)^2 + d(v)^2]
\]

and

\[
Z_2(G) = \sum_{uv \in E(G)} d(u)d(v)
\]

where \( d(u) \) is the degree of a vertex \( u \) in \( G \). The forgotten index of \( G \) is defined as [4, 5]

\[
F(G) = \sum_{uv \in E(G)} [d(u)^2 + d(v)^2].
\]

If diameter of \( G \) is \( \text{diam}(G) \leq 2 \), then [2]

\[
S_1(G) = 4m(n-1) - Z_1(G)
\]

and

\[
S_2(G) = 4m(n-1)^2 - 2(n-1)Z_1(G) + Z_2(G).
\]

2. On eigenvalues of \( S_a(G) \)

Lemma 2.1. Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Then the eigenvalues \( x_1, x_2, \ldots, x_n \) of \( S_a(G) \) satisfies

\[
\sum_{i=1}^{n} x_i = 0
\]

and

\[
\sum_{i=1}^{n} x_i^2 = 2 \sum_{uv \in E(G)} [\sigma(u) + \sigma(v)]^2 = 2M.
\]

Proof: \( \sum_{i=1}^{n} x_i = \text{trace}[S_a(G)] = \sum_{i=1}^{n} s_{ii} = 0. \)

\[
\sum_{i=1}^{n} x_i^2 = \text{trace}[S_a(G)^2] = 2 \sum_{uv \in E(G)} [\sigma(u) + \sigma(v)]^2.
\]

Lemma 2.2. Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Let \( x_1, x_2, \ldots, x_n \) be the eigenvalues of \( S_a(G) \). If \( \text{diam}(G) \leq 2 \), then

\[
\sum_{i=1}^{n} x_i = 0
\]

and

\[
\sum_{i=1}^{n} x_i^2 = 16(n-1)[2m(n-1) - Z_1(G)] + 2F(G) + 4Z_2(G).
\]

Proof: \( \sum_{i=1}^{n} x_i = 0 \) for any vertex \( u \) of \( G \),

\[
\sigma(u) = 2n - 2 - d(u).
\]

By Lemma 2.1,
\[
\sum_{i=1}^{n} x_i^2 = 2 \sum_{u,v \in E(G)} [\sigma(u) + \sigma(v)]^2 = 2 \sum_{u,v \in E(G)} [4n - 4 - d(u) - d(v)]^2
\]
\[
= 2 \sum_{u,v \in E(G)} [(4n - 4) - (4n - 4)(d(u) + d(v)) + (d(u) + d(v))^2]
\]
\[
= 16(n - 1)[2m(n - 1) - Z_1(G)] + 2F(G) + 4Z_2(G).
\]

**Lemma 2.3.** If \((a_1, a_2, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_n)\) are \(n\)-vectors, then Cauchy-Schwartz inequality is
\[
\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \left(\sum_{i=1}^{n} b_i^2\right).
\]
(13)

1. **Bounds for energy of status sum adjacency energy**

The status sum adjacency energy \(S_E(G)\) of a connected graph \(G\) is defined as the sum of the absolute values of the eigenvalues of \(S_d(G)\). That is if \(x_1, x_2, \ldots, x_n\) are the eigenvalues of \(S_d(G)\), then
\[
S_E(G) = \sum_{i=1}^{n} |x_i|.
\]
(14)

The Eq. (14) is analogous to the ordinary graph energy defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of \(G\) [6, 7].

**Theorem 3.1.** Let \(G\) be a connected graph with \(n\) vertices. Then
\[
\sqrt{2M} \leq S_E(G) \leq \sqrt{2nM},
\]
where
\[
M = \sum_{u,v \in E(G)} [\sigma(u) + \sigma(v)]^2.
\]

**Proof.** Upper bound: Choosing \(a_i = 1\) and \(b_i = |x_i|\) for \(i = 1, 2, \ldots, n\) in Lemma 2.3 we get,
\[
\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \left(\sum_{i=1}^{n} b_i^2\right) = 2nM
\]
\[
(SE_E(G))^2 \leq 2nM
\]
\[
S_E(G) \leq \sqrt{2nM}.
\]
Lowe bound:
\[
(SE_E(G))^2 = \left(\sum_{i=1}^{n} |x_i|\right)^2 \geq \sum_{i=1}^{n} x_i^2 = 2M.
\]
Therefore
\[
S_E(G) \geq \sqrt{2M}.
\]

**Corollary 3.2.** Let \(G\) be a graph with \(n\) vertices and \(m\) edges. If \(\text{diam}(G) \leq 2\), then
\[
S_E(G) \geq \sqrt{16(n - 1)[2m(n - 1) - Z_1(G)] + 2F(G) + 4Z_2(G)}.
\]

**Proof.** If \(\text{diam}(G) \leq 2\), then by Lemma 2.2,
\[
2M = \sum_{i=1}^{n} x_i^2 = 16(n - 1)[2m(n - 1) - Z_1(G)] + 2F(G) + 4Z_2(G).
\]

Therefore by Theorem 3.1, result follows.

**Theorem 3.3.** Let \(G\) be a connected graph with \(n\) vertices. Then
\[
S_E(G) \geq \sqrt{2M + n(n - 1) - \text{det}(S_d(G))^{1/n}}
\]
where
\[
M = \sum_{u,v \in E(G)} [\sigma(u) + \sigma(v)]^2.
\]

**Proof.** We follow the same procedure used in [8]. Consider
\[
(SE_E(G))^2 = \left(\sum_{i=1}^{n} |x_i|\right)^2
\]
\[
= \sum_{i=1}^{n} x_i^2 + 2 \sum_{i<j} |x_i| |x_j|
\]
\[
= 2M + \sum_{i<j} |x_i| |x_j|
\]
(15)

As the arithmetic mean of a set of positive number is greater than or equal to their geometric mean, we have
\[
\frac{1}{n(n-1)} \sum_{i<j} |x_i| |x_j| \geq \left[\prod_{i<j} |x_i| |x_j|\right]^{1/(n(n-1))}
\]
\[
= \left[\left(\prod_{i<j} |x_i| |x_j|\right)^{n(n-1)}\right]^{1/(n(n-1))}
\]
\[
= \left[\prod_{i<j} |x_i| |x_j|\right]^{2/n^n} \geq |\text{det}(S_d(G))|^{1/n}.
\]
(16)

From Eqs. (15) and (16) we get,
\[
S_E(G) \geq \sqrt{2M + n(n - 1) - |\text{det}(S_d(G))|^{2/n}}.
\]

3. **Conclusion**

In this work we have obtained some properties of the eigenvalues of the status sum adjacency matrix and bounds for the status sum adjacency energy.

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